

Quantum Information Entropies for the ℓ -state Pöschl-Teller-type potential

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Abstract

In this study, we obtained the position-momentum uncertainties and some uncertainty relations for the Pöschl-Teller-type potential for any ℓ . The radial expectation values of r^{-2} , r^2 and p^2 are obtained from which the Heisenberg Uncertainty principle holds for the potential model under consideration. The Fisher information is then obtained and it is observed that the Fisher-information-based uncertainty relation and the Cramer-Rao inequality hold for this even power potential. Some numerical and graphical results are displayed.

Keywords: Pöschl-Teller-type potential, Schrödinger equation, Uncertainty relations, Fisher information, Cramer-Rao inequality.

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1 Introduction

The quantum mechanical uncertainty principle, first formulated in terms of the standard deviations of the position and momentum probability densities which characterize the quantum-mechanical states of one-dimensional single particle systems, is fundamental to understanding the electronic structure and properties of atoms and molecules [1]. In particular, we have the Heisenberg uncertainty principle [2] for the product of the uncertainties in position and momentum, expressed in terms of Planck's constant [3,4]. For a one-dimensional system defined over $-\infty \leq x \leq \infty$, it is given by the product of the corresponding uncertainties

$$(\Delta x) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \text{ and } (\Delta p) = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}, \quad (1)$$

according to

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}. \quad (2)$$

The Fisher information entropy is defined in position space as [5,6]

$$I[\rho] = \int \frac{[\nabla \rho(\mathbf{r})]^2}{\rho(\mathbf{r})} d\mathbf{r}, \quad (3)$$

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and the corresponding momentum space measure is given by [6]

$$I[\gamma] = \int \frac{[\nabla\gamma(\mathbf{p})]^2}{\gamma(\mathbf{p})} d\mathbf{p}, \quad (4)$$

where ρ and γ are the probability densities in the position and momentum spaces, respectively. The importance of Fisher information as a measure of the information in a distribution is well known. It has many implications in estimation theory, as exemplified by the Cramer-Rao bound which is a fundamental limit on the variance of an estimator [7]. It is a useful tool for characterizing complex signals or systems with applications in geophysics, biology and so on, see refs. [7, 8] for details.

A sharp (smooth) and strongly localized (well spread out) probability density gives rise to a larger(smaller) value of the Fisher information in the position space [6]. The Fisher information entropy in the position space measures the narrowness and the oscillation nature of the probability distribution [9].

The individual Fisher measures are bounded through the Cramer-Rao inequality [10] according to

$$I[\rho] \geq \frac{1}{\sigma^2[\rho]}, \quad I[\gamma] \geq \frac{1}{\sigma^2[\gamma]}, \quad (5)$$

where $\sigma^2[\rho]$ and $\sigma^2[\gamma]$ are, respectively, the standard deviation in the position and momentum spaces [3, 11]. Different properties of some quantum mechanical potentials have been studied using the Heisenberg uncertainty principle [12]. Also, Dehesa et al. (2006) studied the information-theoretic measures for Morse and Pöschl-Teller potentials. It was observed that the ground state of these potentials saturates all the uncertainty relations in an appropriate limit of the parameter [13].

In this study, we shall study the Heisenberg uncertainty principle and the quantum information entropies for the Pöschl-Teller-type potential for any ℓ . As earlier stated, for a particle moving non-relativistically in a central potential $V(r)$, the following uncertainty relation holds ($\langle x \rangle = \langle p \rangle = 0$) [14–21]:

$$\langle r^2 \rangle_{n\ell} \langle p^2 \rangle_{n\ell} \geq \frac{9}{4} \hbar^2. \quad (6)$$

The inequality above becomes equality for the ground state of the harmonic oscillator (HO). The extent to which the above inequality is saturated depends on the potential shape and the state (n, ℓ) of interest. For harmonic oscillator, the inequality in (6) attains its minimum value 2.25 for $n = \ell = 0$ and $\hbar = 1$. Only a few of these central potentials can be solved analytically for the above inequality for any bound state n, ℓ .

The Pöschl-Teller-type potential considered in this study is [22]:

$$V(r) = \frac{\hbar^2 \alpha^2 \lambda(\lambda + 1)}{2M} \tanh^2(\alpha r), \quad (-\infty < r < \infty), \quad (7)$$

where μ is the mass of the particle, and λ denotes the potential depth and α is related to the range of the potential. This potential is a non-homogeneous potential, as described by Sen and Katriel (2006) [23]. It belongs to a class of even-power series potentials, this class of potentials behave like a harmonic oscillator potential (near the origin), often termed ‘oscillator-like’ potentials [20, 21, 25].

The study is organized as follows: In Section 2, we obtain the expectation values of r^{-2} , r^2 and p^2 from which the Heisenberg uncertainty product is obtained for the Pöschl-Teller-type potential for any ℓ . Section 3 contains the Fisher information measure and the Cramer-Rao product of the Pöschl-Teller-type potential for any ℓ . The conclusion is given in Section 4.

2 Position-momentum uncertainty relations

The energy eigenvalues for the ℓ -state Pöschl-Teller-type potential is obtained as [24]

$$E_{n\ell} = \frac{\hbar^2}{2\mu} \left[4d_0\Lambda - 4n^2 - 4n + 8n\gamma + 4\gamma - 8n\zeta - 4\zeta - \frac{3}{2} - \Lambda + 8\gamma\zeta \right], \quad (8)$$

where

$$\Lambda = \alpha^2 \ell(\ell + 1), \quad \beta = \alpha^2 \lambda(\lambda + 1), \quad s = \sinh^2(\alpha r), \quad \gamma = \sqrt{\frac{1}{16} + \frac{\beta}{4}}, \quad \zeta = \sqrt{\frac{1}{16} + \frac{\Lambda}{4}}. \quad (9)$$

Also, $n = 0, 1, 2, \dots, [\lambda]$, where $[\lambda]$ denotes the largest integer inferior to λ . The wave function is obtained as [24]

$$R_{n\ell} = s^{\frac{1}{4}+\zeta} (1+s)^{\frac{1}{4}-\gamma} P_n^{(2\zeta, -2\gamma)}(1+2s). \quad (10)$$

The expectation value $\langle r^{-2} \rangle$ of the Pöschl-Teller-type potential is calculated explicitly, by using the Hellmann-Feynman theorem (HFT) [20, 21, 26–34]. The HFT states that a non-degenerate eigenvalue $E(q)$ of a parameter-dependent Hermitian operator $H(q)$, the associated eigenvector $\Psi(q)$, changes with respect to the parameter q according to the formula [33, 34]:

$$\frac{\partial E_{n,\ell}(q)}{\partial q} = \left\langle \Psi_{n,\ell}(q) \left| \frac{\partial H(q)}{\partial q} \right| \Psi_{n,\ell}(q) \right\rangle. \quad (11)$$

The Hamiltonian of the system is

$$H = \frac{p^2}{2\mu} + \frac{\hbar^2 \alpha^2 \ell(\ell + 1)}{2\mu r^2} + \frac{\hbar^2 \alpha^2 \lambda(\lambda + 1)}{2\mu r^2} \tanh^2 \alpha r \quad (12)$$

and by putting $q = \ell$, we obtain

$$\langle r^{-2} \rangle_{n\ell} = -\frac{2}{3} - \frac{n}{\sqrt{\zeta}} - \frac{1}{2\sqrt{\zeta}} + \frac{\gamma}{\sqrt{\zeta}}. \quad (13)$$

The graph of $\langle r^{-2} \rangle_{n\ell}$ against λ for some values of n and ℓ is plotted in Figure 1. We observe that $\langle r^{-2} \rangle_{n\ell}$ increases linearly with λ . The values are also shown in Table 1 for some n and ℓ . It is observed from Table 1 that $\langle r^{-2} \rangle_{n\ell}$ decreases with increasing n when ℓ is fixed.

In order to obtain the uncertainty product $(\Delta r)_{n\ell}^2 (\Delta p)_{n\ell}^2$, we need to calculate the following expectation values: $\langle r \rangle_{n\ell}$, $\langle p \rangle_{n\ell}$, $\langle r^2 \rangle_{n\ell}$ and $\langle p^2 \rangle_{n\ell}$. As expected for a particle moving in the symmetric potential well, both $\langle r \rangle_{n\ell}$ and $\langle p \rangle_{n\ell}$ are zero [22]. $\langle r^2 \rangle_{n\ell}$ is obtained as

$$\begin{aligned} \langle r^2 \rangle_{n\ell} &= 4\pi \int_{-\infty}^{\infty} r^2 R_{n\ell}^*(r) r^2 R_{n\ell}(r) dr \\ &= 4\pi N^2 \int_0^1 \frac{(\sinh^{-1} \sqrt{s})^4}{2\alpha^5} s^{2\zeta} (1+s)^{-2\gamma} [P_n^{(2\zeta, -2\gamma)}(1+2s)]^2 ds. \end{aligned} \quad (14)$$

From the Hamiltonian of the system, we obtain

$$\begin{aligned}\langle p^2 \rangle_{n\ell} &= 4\pi \int_{-\infty}^{\infty} r^2 R_{n\ell}^*(r) \left[2\mu H - \frac{\hbar^2 \Lambda}{r^2} - \hbar^2 \beta \tanh^2 \alpha r \right] R_{n\ell}(r) dr \\ &= 4\pi \left[2\mu E_{n,\ell} - \hbar^2 \Lambda \int_{-\infty}^{\infty} R_{n\ell}^2(r) dr - \hbar^2 \beta \int_{-\infty}^{\infty} r^2 \tanh^2 \alpha r R_{n\ell}^2(r) dr \right],\end{aligned}\quad (15)$$

so that

$$\begin{aligned}\langle p^2 \rangle_{n\ell} &= 4\pi \left\{ 2\mu E_{n,\ell} - \frac{\hbar^2 \Lambda N^2}{2\alpha} \int_0^1 s^{2\zeta} (1+s)^{-2\gamma} [P_n^{(2\zeta, -2\gamma)}(1+2s)]^2 ds \right. \\ &\quad \left. - \frac{\hbar^2 \beta N^2}{2\alpha^3} \int_0^1 s^{1+2\zeta} (1+s)^{-1-2\gamma} (\sinh^{-1} \sqrt{s})^2 [P_n^{(2\zeta, -2\gamma)}(1+2s)]^2 ds \right\}.\end{aligned}\quad (16)$$

A state is defined to be squeezed if $(\Delta r)^2 < 0.5$. Our results from Tables 2-6 show a squeezed phenomenon in the position r for the ground state when $\lambda \geq 2.5$. For the first excited state ($n = 1$), we have squeezed phenomena from $\lambda = 2.2$ to $\lambda = 5.3$, also at $\lambda \geq 11.4$. We also have squeezed phenomena at various points for the states ($n = 2, \ell = 0$), ($n = 2, \ell = 1$), ($n = 3, \ell = 2$). It is observed from Tables 2 - 6 that the lower bounds for the single particle in any central potential is obtained as [8, 11]

$$(\Delta r)_{n\ell}^2 (\Delta p)_{n\ell}^2 = \langle r^2 \rangle_{n\ell} \langle p^2 \rangle_{n\ell} \geq \left(\ell + \frac{3}{2} \right)^2 \quad (17)$$

holds for this potential. The least value for $\ell = 0$ with various values of n obtained is 3.306209 which is greater than the expected minimum value (2.250) as shown in Tables 2 - 4. For various values of n and ℓ , the Heisenberg uncertainty principle holds for the various values of λ considered as it can be seen from Tables 5 - 9, since the least value obtained is greater than the minimum value of equation (17).

3 Fisher Information

The Fisher information of the Pöschl-Teller-type potential for any ℓ . Noting that the Fisher information for the expectation values of r and p can be obtained by using the following relations: [3]

$$I[\rho] = 4 \langle p^2 \rangle_{n\ell} - 2(2\ell + 1)|m| \langle r^{-2} \rangle_{n\ell} \quad (18)$$

and

$$I[\gamma] = 4 \langle r^2 \rangle_{n\ell} - 2(2\ell + 1)|m| \langle p^{-2} \rangle_{n\ell}. \quad (19)$$

The $\langle r^2 \rangle$ and $\langle p^2 \rangle$ have been obtained in equations (14) and (16), respectively. On substituting equations (14) and (16) into equations (18) and (19), the Fisher information in the position and momentum spaces are obtained. In this case, the magnetic quantum number (m) is zero. According to Dehesa et al. (2007), the product of both equations above can not be less than 36.00 [35]:

$$I[\rho] \times I[\gamma] \geq 36.00. \quad (20)$$

The values obtained are shown in Tables 10 - 14. The Fisher-information-based uncertainty relation holds for this potential model, since the minimum value of $I[\rho] \times I[\gamma]$ from the

Tables is 52.89935 which is greater than the expected minimum value of 36 (from equation (20)). Similarly, the Cramer-Rao inequality given by Dehesa et al. (2007) [35]

$$I[\rho] \times V[\rho] \geq D^2 \quad (21)$$

is satisfied for this potential as shown in Table 15. In this case, D is 3, so that the minimum value is 9.00.

4 Conclusion

We have studied some entropic-uncertainty relations for the Pöschl-Teller-type potential for any ℓ . We obtained the expectation values of r^{-2} , r^2 and p^2 . The Heisenberg Uncertainty principle for any ℓ is satisfied for this potential model as seen in the numerical results. The Fisher information is then obtained by making use of the expectation values of r^2 and p^2 , coupled with the fact that the magnetic quantum number m is zero in the case considered. The validity of the Heisenberg, Fisher-information-based uncertainty relations and Cramer-Rao products for the Pöschl-Teller type has been investigated. We established that the condition of the Fisher-Information-based uncertainty product is satisfied for the Pöschl-Teller type potential model. Also, the Cramer-Rao inequality holds for this potential model.

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Table 1: Numerical results of $\langle r^{-2} \rangle_{n\ell}$ for the Pöschl-Teller-type potential with $\lambda = 100$, $\hbar = 2\mu = 1$, $\alpha = 0.1$ for some n and ℓ

n	ℓ	$\langle r^{-2} \rangle_{n\ell}$
0	0	8.39564
1	0	6.39564
2	0	4.39564
3	0	2.39564
1	2	6.02588
2	2	4.13059
3	2	2.23530
4	2	0.34002
1	5	5.13217
2	5	3.48998
3	5	1.84778
4	5	0.20559

Table 2: Numerical results for the uncertainty relation $(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$ with $\hbar = 2\mu = 1$, $\alpha = 1$, $n = 0$ and $\ell = 0$

λ	$\langle r^2 \rangle_{n\ell}$	$\langle p^2 \rangle_{n\ell}$	$(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$	$\min ((\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2)$
0.5	0.551859	5.99104	3.306209	2.25
1.5	0.526554	42.5751	22.41728	2.25
2.5	0.498741	78.5373	39.16977	2.25
3.5	0.468876	113.996	53.44999	2.25
4.5	0.437563	149.079	65.23145	2.25
5.5	0.405598	183.915	74.59556	2.25
6.5	0.373856	218.619	81.73202	2.25
7.5	0.343174	253.286	86.92117	2.25
8.5	0.314248	287.981	90.49745	2.25
9.5	0.287563	322.742	92.80866	2.25

Table 3: Numerical results for the uncertainty relation $(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$ with $\hbar = 2\mu = 1$, $\alpha = 1$, $n = 1$ and $\ell = 0$

λ	$\langle r^2 \rangle_{n\ell}$	$\langle p^2 \rangle_{n\ell}$	$(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$	$\min ((\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2)$
1.5	0.548912	17.3952	9.548434	2.25
2.5	0.461496	103.858	47.93005	2.25
3.5	0.302590	190.969	57.78531	2.25
4.5	0.334296	276.538	92.44555	2.25
5.5	0.516568	357.880	184.8694	2.25
6.5	0.569420	439.477	250.2470	2.25
7.5	0.572668	521.226	298.4895	2.25
8.5	0.560332	602.821	337.7799	2.25
9.5	0.541791	684.221	370.7048	2.25

Table 4: Numerical results for the uncertainty relation $(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$ with $\hbar = 2\mu = 1$, $\alpha = 1$, $n = 2$ and $\ell = 0$

λ	$\langle r^2 \rangle_{n\ell}$	$\langle p^2 \rangle_{n\ell}$	$(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$	$\min ((\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2)$
2.5	0.550407	28.0161	15.4203	2.25
3.5	0.362748	165.251	59.9445	2.25
4.5	0.411899	300.560	123.800	2.25
5.5	0.561492	432.187	242.670	2.25
6.5	0.540564	565.700	305.797	2.25
7.5	0.476402	700.333	333.640	2.25
8.5	0.396338	836.261	331.442	2.25
9.5	0.369105	970.598	358.253	2.25

Table 5: Numerical results for the uncertainty relation $(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$ with $\hbar = 2\mu = 1$, $\alpha = 1$, $n = 2$ and $\ell = 1$

λ	$\langle r^2 \rangle_{n\ell}$	$\langle p^2 \rangle_{n\ell}$	$(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$	$\min ((\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2)$
3.5	0.578148	117.507	67.93561	6.25
4.5	0.484400	277.361	134.3537	6.25
5.5	0.317118	437.433	138.7170	6.25
6.5	0.418443	595.168	249.0439	6.25
7.5	0.534586	751.435	401.7066	6.25
8.5	0.538793	908.354	489.4148	6.25
9.5	0.498933	1066.09	531.9075	6.25
10.5	0.437253	1225.02	535.6437	6.25

Table 6: Numerical results for the uncertainty relation $(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$ with $\hbar = 2\mu = 1$, $\alpha = 1$, $n = 3$ and $\ell = 2$

λ	$\langle r^2 \rangle_{n\ell}$	$\langle p^2 \rangle_{n\ell}$	$(\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2$	$\min ((\Delta r)_{n\ell}^2(\Delta p)_{n\ell}^2)$
4.5	0.639235	61.0028	38.9951	12.25
5.5	0.596992	294.683	175.923	12.25
6.5	0.503234	527.632	265.522	12.25
7.5	0.340425	759.800	258.655	12.25
8.5	0.449519	990.761	445.366	12.25
9.5	0.528862	1221.84	646.185	12.25
10.5	0.503222	1453.55	731.458	12.25

Table 7: The values of $(\Delta r)_{n\ell}$ for some λ with $\hbar = 2\mu = 1$ and $\alpha = 1$

n	ℓ	$\lambda = 20$	$\lambda = 50$	$\lambda = 100$	$\lambda = 200$
3	0	0.750849	0.508588	0.347300	0.241542
4	0	0.663283	0.583597	0.394231	0.272851
5	0	0.706252	0.654627	0.437750	0.301451
6	1	0.678838	0.739476	0.497916	0.340207
7	1	0.707238	0.751129	0.536695	0.364714
8	1	0.716054	0.690046	0.574314	0.388097

Table 8: The values of $(\Delta p)_{n\ell}$ for some λ with $\hbar = 2\mu = 1$ and $\alpha = 1$

n	ℓ	$\lambda = 20$	$\lambda = 50$	$\lambda = 100$	$\lambda = 200$
3	0	54.6780	90.8740	130.833	186.646
4	0	60.1654	101.147	146.469	209.523
5	0	64.1146	110.033	160.284	229.923
6	1	68.4516	121.564	178.512	257.144
7	1	70.2057	128.531	189.370	273.577
8	1	71.2527	135.263	199.394	288.908

Table 9: The values of $(\Delta r \Delta p)_{n\ell}$ for some λ with $\hbar = 2\mu = 1$ and $\alpha = 1$

n	ℓ	$\lambda = 20$	$\lambda = 50$	$\lambda = 100$	$\lambda = 200$	$\min(\Delta r \Delta p)_{n\ell}$
3	0	41.0549	46.2170	45.4383	45.0828	1.50
4	0	39.9067	59.0290	57.7426	57.1686	1.50
5	0	45.2811	72.0306	70.1643	69.3105	1.50
6	1	46.4675	89.8937	88.8840	87.4822	2.50
7	1	49.6521	96.5434	101.634	99.7770	2.50
8	1	51.0208	93.3377	114.515	112.124	2.50

Table 10: Fisher information measure with $2\mu = \hbar = \alpha = 1$, $n = 0$ and $\ell = 0$

λ	$I[\rho]$	$I[\gamma]$	$I[\rho]I[\gamma]$	$\min (I[\rho]I[\gamma])$
0.5	23.96416	2.207436	52.89935	36.00
1.5	170.3004	2.106140	358.6765	36.00
2.5	314.1492	1.994964	626.7163	36.00
3.5	455.9840	1.875504	855.1998	36.00
4.5	596.3160	1.750252	1043.703	36.00
5.5	735.6600	1.622392	1193.529	36.00
6.5	874.4760	1.495424	1307.712	36.00
7.5	1013.144	1.372696	1390.739	36.00
8.5	1151.924	1.256992	1447.959	36.00
9.5	1290.968	1.150252	1484.939	36.00

Table 11: Fisher information measure with $2\mu = \hbar = \alpha = 1$, $n = 1$ and $\ell = 0$

λ	$I[\rho]$	$I[\gamma]$	$I[\rho]I[\gamma]$	$\min (I[\rho]I[\gamma])$
1.5	69.58080	2.195648	152.7749	36.00
2.5	415.4302	1.845984	766.8808	36.00
3.5	763.8760	1.210360	924.5650	36.00
4.5	1106.152	1.337184	1479.129	36.00
5.5	1431.520	2.066272	2957.910	36.00
6.5	1757.908	2.277680	4003.958	36.00
7.5	2084.904	2.290672	4775.831	36.00
8.5	2411.284	2.241328	5404.478	36.00
9.5	2736.884	2.167164	5931.276	36.00

Table 12: Fisher information measure with $2\mu = \hbar = \alpha = 1, n = 2$ and $\ell = 0$

λ	$I[\rho]$	$I[\gamma]$	$I[\rho]I[\gamma]$	$\min (I[\rho]I[\gamma])$
2.5	112.0644	2.201628	246.7241	36.00
3.5	661.0040	1.450992	959.1115	36.00
4.5	1202.240	1.647596	1980.806	36.00
5.5	1728.748	2.245968	3882.713	36.00
6.5	2262.800	2.162256	4892.753	36.00
7.5	2801.332	1.905608	5338.241	36.00
8.5	3345.044	1.585352	5303.072	36.00
9.5	3882.392	1.476420	5732.041	36.00

Table 13: Fisher information measure with $2\mu = \hbar = \alpha = 1, n = 2$ and $\ell = 1$

λ	$I[\rho]$	$I[\gamma]$	$I[\rho]I[\gamma]$	$\min (I[\rho]I[\gamma])$
3.5	470.0280	2.312564	1086.970	36.00
4.5	1109.444	1.937600	2149.659	36.00
5.5	1749.732	1.268472	2219.486	36.00
6.5	2380.672	1.673772	3984.702	36.00
7.5	3005.740	2.138344	6427.306	36.00
8.5	3633.416	2.155172	7830.636	36.00
9.5	4264.360	1.995732	8510.520	36.00
10.5	4900.080	1.749012	8570.299	36.00

Table 14: Fisher information measure with $2\mu = \hbar = \alpha = 1, n = 3$ and $\ell = 2$

λ	$I[\rho]$	$I[\gamma]$	$I[\rho]I[\gamma]$	$\min (I[\rho]I[\gamma])$
4.5	244.0112	2.556940	623.9220	36.00
5.5	1178.732	2.387968	2814.774	36.00
6.5	2110.528	2.012936	4248.479	36.00
7.5	3039.200	1.361700	4138.479	36.00
8.5	3963.044	1.798076	7125.854	36.00
9.5	4887.360	2.115448	10338.96	36.00
10.5	5814.200	2.012888	11703.33	36.00

Table 15: The values of the Cramer-Rao product $I[\rho]V[\rho]$ with $2\mu = \hbar = \alpha = 1$, $n = 3$ and $\ell = 2$

λ	$n = 0, \ell = 0$	$n = 1, \ell = 0$	$n = 2, \ell = 1$	$n = 3, \ell = 2$	$\min (I[\rho]V[\rho])$
0.5	13.22484				9.00
1.5	89.66912	38.19374			9.00
2.5	156.6791	191.7202			9.00
3.5	213.8000	231.1412	271.7425		9.00
4.5	260.9258	369.7822	537.4147	155.9805	9.00
5.5	298.3822	739.4774	554.8715	703.6936	9.00
6.5	326.9281	1000.988	996.1755	1062.089	9.00
7.5	247.6847	1193.958	1606.827	1034.620	9.00
8.5	361.9898	1351.120	1957.659	781.4640	9.00
9.5	371.2346	1482.819	2127.630	2584.739	9.00

Figure 1: Graph of $\langle r^{-2} \rangle_{n\ell}$ against λ for some n and ℓ

